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A Digital, Noninteger Order, Differentiator using Laguerre Orthogonal Sequences

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***Abstract*—This paper presents a novel procedure to design a digital, noninteger order, differentiator. The method is based on the Laguerre series expansion. Firstly, a discrete equivalent of the noninteger derivative Euler backward operator is given in the *z*-domain. Secondly, this operator is expanded into a Taylor series, which provides the data for the approximation of the Laguerre noninteger order digital derivative operator. Simulation results show the accuracy of the approximation, by measuring the frequency response for different values of the derivative noninteger order.**

Index Terms: Digital differentiators, noninteger (fractional) order derivatives and integrals, fractional order systems, fractional controllers, fractal robustness, Laguerre sequences.

I. INTRODUCTION

FRACTIONAL calculus (FC) is an old issue of the mathematical science, which goes back to the beginning of the differential calculus. It generalizes the derivative operator D^v by encompassing real and complex values for the exponent v, which is ordinarily integer-valued [1], [2]. Only more recently, derivatives of noninteger order have been considered in physics and engineering. Fractional order differential equations (FODE) have been used to get deeper knowledge of properties of materials, physical processes and phenomena [1], [3–5]. Among many other research fields, interesting applications can be found in the theory of viscoelasticity [6], [7], in biology [2] where FODE apparently better describe neuronal behavior [8], and in geophysical data processing [9]. Fractional derivatives also accurately describe problems of bioelectricity, tissue mechanics, and bioengineering [10].

As far as electronic engineering concerns, Bode proposed an ideal shape of the loop transfer function of feedback amplifiers, the "ideal cut-off characteristic" $(s/\omega_{gc})^v$, where s and ω_{gc} are the complex variable and the gain crossover frequency. The choice of such transfer function, indeed, gives a closed loop system that it is insensitive to gain changes. More recently, new applications of fractional calculus are proposed to solve the problems of automatic control [2], [11], circuit theory [12], [13], and dynamical systems [14].

However, to the best author knowledge, the recent applications in automatic control and signal processing have been mostly developed in the frequency domain. Only some algorithms have been developed in the time domain. This domain, indeed, is suited for *Z*-transform analysis and discrete time implementation. In the signal processing literature, there are attempts for designing systems performing digital noninteger order differentiation (DND for brevity). Namely, even if the design of digital differentiators is a classical issue of the signal processing literature [15], [16], there are relatively few contributions dealing with DND. For example, reference [17] provides a convolution-based algorithm for developing DND; the Finite Impulse Response (FIR) approximation of [18], which is obtained by defining the fractional derivative of a power function and then by solving linear equations of the Vandermonde form; and, recently, the FIR system obtained by [19], which provides a DND based on the Newton series.

In the area of automatic control, the direct discretization method includes the direct power series expansion of the Euler and Tustin operators, and the continuous fraction expansion [9], [13], [20]–[21].

This paper introduces a Laguerre filter realization of a DND. The starting point is to give a discrete equivalent of the noninteger derivative Euler backward operator in the *z*-domain. Then, the Taylor expansion of this operator leads to a power series in z^{-1} and generates a sequence of samples, satisfying the conditions necessary to synthesize a discrete Laguerre operator of a DND. This step is useful for two main reasons.

Firstly, using Laguerre sequences leads to a data compression, providing a parsimonious expansion. Namely, it is advantageous to approximate the differentiator with a limited number of terms, because of the limits in the memory at disposal of microprocessor systems.

Secondly, using the power series expansion leads to discrete transfer functions having the form of polynomials. This structure is not convenient for control purpose. Namely, approximations based on rational transfer functions are more appropriate.

This paper is organized as follows. Section II shows how to generalize the digital derivative Euler backward operator to consider a noninteger order of differentiation, v. Section III explains how to use a finite number of Laguerre sequences to obtain a good approximation of the infinite Taylor series

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expansion of the DND. Section IV reports the results of simulation experiments to test the frequency response of the proposed Laguerre digital differentiators, for several values of the noninteger order ν. Section V is a brief discussion about future real applications of DND in control systems. Section VI gives the conclusion.

II. THE EULER NONINTEGER ORDER DIGITAL DIFFERENTIATOR

Simply speaking, a first derivative digital differentiator (DD) performs differentiation by processing the sequence of an input signal, $x(kT)$ ($k = 0, 1, 2, \ldots$) at sampling period T, to generate a response, $y(kT)$. The well-known Euler DD is based on the first order backward Taylor series expansion and has $g_E(z) = (1-z^{-1})/T$ as Z-transform. Hence:

$$y(z) = g_E(z)\,x(z) \tag{1}$$

where $x(z)$ and $y(z)$ are the Z-transforms of the input and output signals.

Now, consider the Z-transform of the Grunwald-Letnikov [1], [2] approximation of a derivative of fractional order ν of the input sequence:

$$Z\left\{D^{\nu}x(kT)\right\} = \left\{\frac{1}{T^{\nu}}\sum_{k=0}^{\infty}(-1)^{k}\,B_k^{\nu}\,z^{-k}\right\}x(z) \tag{2}$$

which approximates the time increment in the discretization with the sampling period, T.

In (2), B_k^{ν} is the generalized binomial coefficient, defined by the finite product [22]:

$$B_k^{\nu} = \binom{\nu}{k} = \prod_{j=1}^{k}\left(1+\frac{\nu-k}{j}\right) = \prod_{j=0}^{k-1}\left(\frac{\nu-j}{k-j}\right) \text{ with } \binom{\nu}{0} = 1 . \tag{3}$$

Note that the binomial coefficients can be easily computed by using [22]:

$$B_{k+1}^{\nu} = \frac{\nu-k}{k+1}B_k^{\nu} \tag{4}$$

and that the position $C_k^{\nu} = (-1)^{k}\,B_k^{\nu}$ with $C_0^{\nu} = 1$ leads to

$$C_k^{\nu} = (-1)^{k}\,\frac{\nu-k+1}{k}B_{k-1}^{\nu} = \left(1-\frac{1+\nu}{k}\right)C_{k-1}^{\nu} . \tag{5}$$

Then, the coefficients C_k^{ν} can be recursively determined by using (5). At this point, the bracketed expression of (2) becomes [2]:

$$\left\{\frac{1}{T^{\nu}}\sum_{k=0}^{\infty}C_k^{\nu}\,z^{-k}\right\} = \left(\frac{1-z^{-1}}{T}\right)^{\nu} = \psi_{\nu}(z) . \tag{6}$$

Hence, the function $\psi_{\nu}(z)$ yields the operator approximating the ν-order derivative in the z-domain. It can be considered a generalization of the Euler (or first back difference) operator to noninteger exponent ν.

Theoretically, it requires an infinite number of terms and has implicitly ‘a memory’ of past input samples. In practice, however, the N-terms truncated expression of the Taylor series expansion in (6) can be used to approximate the ν-th derivative.

This approximation of $\psi_{\nu}(z)$ is in the form of polynomials and often requires many coefficients C_k^{ν} for obtaining good results.

This motivates the use of the Laguerre expansion to get an accurate approximation of the ν-order derivative operator with few parameters.

III. THE LAGUERRE NONINTEGER ORDER DIGITAL DIFFERENTIATOR

The Laguerre sequences [23]

$$\lambda_m(k,b) = \sqrt{1-b^2}\sum_{j=0}^{m}(-1)^{m+j}\,\frac{(k+m-j)!}{j!\,(m-j)!\,(k-j)!}\,b^{k+m-2j} \tag{7}$$

form a complete orthonormal set in the space ℓ^2 of finite energy causal frequencies for any fixed value of the parameter $0 < b < 1$. They are obtained by using the Gram-Schmidt orthogonalization on the sequences $k^{\nu}b^{k}$ [24]. The Z-transform of the sequence (7) is given by:

$$\Lambda_m(z,b) = \sqrt{1-b^2}\,\frac{(z^{-1}-b)^{m}}{(1-bz^{-1})^{m+1}} \tag{8}$$

where the following recurrence relation

$$\Lambda_{m+1}(z,b) = A(z,b)\,\Lambda_m(z,b) = [A(z,b)]^{m+1}\,\Lambda_0(z,b) \tag{9}$$

holds true, for $m = 0, 1, \ldots$, with

$$A(z,b) = \frac{z^{-1}-b}{1-bz^{-1}} \text{ and } \Lambda_0(z,b) = \frac{\sqrt{1-b^2}}{1-bz^{-1}} . \tag{10}$$

Now, the ratio test [24] shows that the power series on the left member of (6) is absolutely convergent in an open circle of radius 1. Hence, there exists an expansion of $\psi_{\nu}(k)$ in terms of Laguerre sequences:

$$\psi_v(k,b) = \sum_{m=0}^{\infty} a_m \, \lambda_m(k,b) \tag{11}$$

where

$$a_m = \frac{1}{T^v} \sum_{k=0}^{\infty} C_k^v \, \lambda_m(k,b) \tag{12}$$

is the expression of the Laguerre coefficients. E.g., (11) uses an infinite number of Laguerre sequences. In practice, it is possible to approximate $\psi_v(k)$ with a finite number of Laguerre sequences and an N-terms truncated expression. Hence, (8) and (10) lead to:

$$\psi_v(z,b) = \frac{\sqrt{1-b^2}}{1-bz^{-1}} \sum_{m=0}^{N-1} a_m \left(\frac{z^{-1}-b}{1-bz^{-1}} \right)^m \tag{13}$$

which is the expression of the DND Laguerre operator.

The filter of Fig. 1 gives the well-known structure of (13), which for $b = 0$ simplifies to the familiar tapped-delay line. Note that the value of the Laguerre parameter b influences the quality of the DND Laguerre operator. Hence, the selection of the appropriate value of b is an important problem [25]–[27]. According to [26], for a given N, choosing the Laguerre parameter as one of the roots of the equations $a_{N-1} = 0$ and $a_N = 0$ minimizes the squared error related to the approximation. Clearly, for each value of v an appropriate value of b must be determined.

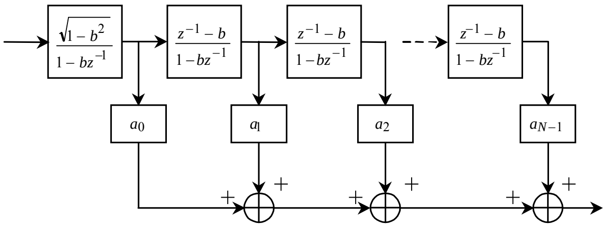

Fig. 1. The Laguerre network.

IV. SIMULATION EXPERIMENTS

An example shows the efficiency of the approximation. The following coefficients result, by putting $T = 0.05$ and $v = 0.5$, with $b = 0.569$:

$a_0 = 2.414 \quad a_1 = -3.268 \quad a_2 = 2.194 \quad a_3 = -0.160$
$a_4 = 0.961 \quad a_5 = -0.657 \quad a_6 = 0.368 \quad a_7 = -0.255$
$a_8 = 0.130 \quad a_9 = -0.100 \quad a_{10} = 0.041 \quad a_{11} = -0.040$
$a_{12} = 0.010 \quad a_{13} = -0.020 \quad a_{14} = 0.000.$

Substituting $z = e^{j\omega t}$ in (12) leads to the frequency response of the Laguerre DND. Figure 2 shows the phase diagrams for different values of the noninteger order of differentiation, v. Note that in a wide frequency range the phase curve of the DND is nearly flat and that the lead approximates $0.5v\pi$, as it is expected. The dotted lines yield the phase of $\psi_v(e^{j\omega T})$ which coincides with the high frequency part of the Laguerre DND. Figure 3 gives the magnitude responses of the DND for different values of v. The dotted curves give the magnitude of the operator $\psi_v(e^{j\omega T})$. The solid curves representing the magnitude of the Laguerre DND coincide in the high frequency part with the dotted graph. Both Figs. 2 and 3 show that there are relatively few terms sufficient to have an excellent approximation of a derivative action in a wide frequency range.

V. DND IN CONTROL APPLICATIONS: SOME PERSPECTIVES

DND are important building blocks of fractional-order controllers (FOC), which recently have received increasing attention [20], [28]. This interest is due to two main reasons. First of all, dynamical systems that are modeled by FODE need FOC for a more effective control [29]. A typical case-study, proposed in [30], implements a digital fractional PD^v controller for an electric radiator, modeled by a fractional-order transfer function. A second reason for the attention to FOC, is due to the effectiveness of the frequency analysis, which enlightens the flexibility of digital FOC in controlling integer-order dynamical systems. E.g., by Figs. 2 and 3 it is clear that, by changing the fractional order v of the DND, it is possible to directly shape the frequency response of the whole controlled system, providing a more straightforward design of robust control system against uncertainties. Hence, the number of applications of DND and of digital FOC is supposed to increase not only in robotics and automation applications [31], but also in new fields as biomimetic control, which pursues "*good design from nature*" [32]. In conclusion, the here proposed DND enjoys these characteristics.

However, even if the progress of digital techniques and the increase of processing power promotes the implementation of DND and FOC, their diffusion still encounters some difficulties. Probably, these are due to the fact that, while the frequency characteristics of DND have a direct interpretation, the time domain analysis requires more mathematical skills. Unlike integer-order digital derivative operators, indeed, by (2) and (11) the DND requires an infinite number of terms. I.e., integer derivatives are local operators, while the fractional ones have memory of the past events, hence they are less intuitive operators. However, the development of "ad hoc" design methods for digital FOC can contribute to increase the success of these controllers.

VI. CONCLUSION

This paper gives an approach to design a digital noninteger order differentiator. An efficient Laguerre approximation of the derivative Euler operator is given. Namely, the accuracy of the approximation is demonstrated by simulation in a wide frequency range, for different values of the noninteger order of differentiation, *v*. Both the magnitude and phase Bode diagrams showed that few Laguerre coefficients are necessary to get good results.

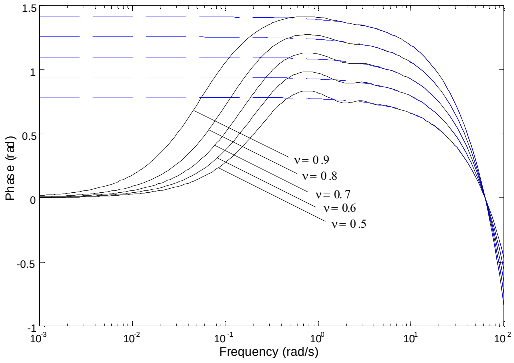

Fig. 2. Phase Bode diagrams of the Laguerre noninteger order digital differentiator.

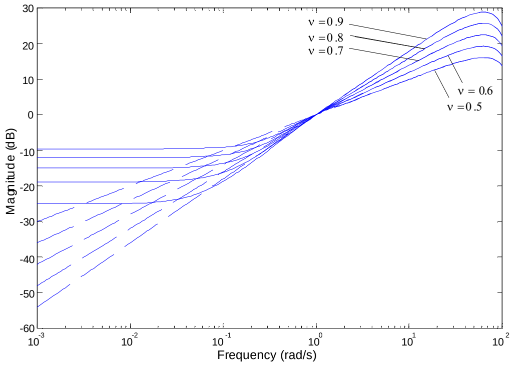

Fig. 3. Magnitude Bode diagrams of the Laguerre noninteger order digital differentiator.

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